Data Science in Aerospace

Random Variables

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1 Inspection

An aviation company has 7 inspectors, 2 of whom are female. For a random inspection of an aircraft, two inspectors are selected at random. Let Y denote the number of female inspectors selected.

- Define the sample space for the experiment, designating the inspectors as A, B, C, D, E (males) and F, G (females).

$$\mathcal{S} = \{\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{A, E\}, \{A, F\}, ...\}$$

- Define the value of the random variable Y for each element of the sample space.

$$Y : \mathcal{S} \to \mathcal{S}' = \{0, 1, 2\}$$

$$\begin{aligned}
&Y(\{F, G\}) = 2 \\
&Y(\{A, F\}, \{B, F\}, \{C, F\}, ...) = 1 \\
&Y(\text{others}) = 0
\end{aligned}$$

- Define the probability and distribution functions of the random variable Y. Represent both functions in tabular form and using bar diagrams.

$$Y : \text{Number of female inspectors selected}$$

- Probability Function

$$P(Y=0) = \frac{C(5,2)}{C(7,2)} = \frac{10}{21}$$
$$P(Y=1) = \frac{C(2,1) \times C(5,1)}{C(7,2)} = \frac{10}{21}$$
$$P(Y=2) = \frac{C(2,2)}{C(7,2)} = \frac{1}{21}$$

- Distribution Function

$$F(0) = \frac{10}{21}$$
$$F(1) = \frac{10}{21} + \frac{10}{21} = \frac{20}{21}$$
$$F(2) = \frac{20}{21} + \frac{1}{21} = 1$$

- Calculate the expected value and standard deviation of the random variable Y.
  - Expected value

$$\begin{aligned} E[Y] &= \sum_i yP(Y=y) \\ &= 0 \times P(Y=0) + 1 \times P(Y=1) + 2 \times P(Y=2) \\ &= 0 \times \frac{10}{21} + 1 \times \frac{10}{21} + 2 \times \frac{1}{21} \\ &\approx 0.57 \end{aligned}$$

  - Variance

$$\begin{aligned} \text{Var}(Y) &= \sum_i (y - E[Y])^2 P(Y=y) \\ &= (0-0.57)^2 \frac{10}{21} + (1-0.57)^2 \frac{10}{21} + (2-0.57)^2 \frac{1}{21} \\ &\approx 0.34 \end{aligned}$$

  - Standard deviation

$$\sqrt{\text{Var}(Y)} \approx 0.58$$

2 Aircraft Maintenance Check

An airport maintenance facility has a fleet of 10 aircraft, 3 of which are fully operational and 7 require maintenance. Three aircraft are selected at random without replacement for a safety inspection.

- What is the probability that at most one fully operational aircraft is selected?

$$P = \frac{C(7,3)}{C(10,3)} + \frac{C(7,2) \times C(3,1)}{C(10,3)} \approx 0.817$$

- If exactly one fully operational aircraft was selected in the three inspections, what is the probability that it was selected during the second inspection?

$$A : \text{Fully operational aircraft in second inspection}$$

$$B : \text{Only one operational aircraft was selected in the three inspections}$$

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{\frac{7 \times 3 \times 6}{P(10,3)}}{\frac{(7 \times 6 \times 3) + (7 \times 3 \times 6) + (3 \times 7 \times 6)}{P(10,3)}} = \frac{1}{3} \end{aligned}$$

- Let X be the random variable representing the number of fully operational aircraft selected in the three inspections. Determine:
  - The probability mass function (PMF) of X.

$$X : \text{Number of fully operational aircraft in three inspections}$$

$$
\begin{aligned}
P(X=0) &= \frac{C(7,3)}{C(10,3)} \approx 0.292 \\
P(X=1) &= \frac{C(3,1)\times C(7,2)}{C(10,3)} \approx 0.525 \\
P(X=2) &= \frac{C(3,2)\times C(7,1)}{C(10,3)} \approx 0.175 \\
P(X=3) &= \frac{C(3,3)}{C(10,3)} \approx 0.008
\end{aligned}
$$

- The cumulative distribution function (CDF) of X.

$$
\begin{aligned}
F(0) &= 0.292 \\
F(1) &= 0.292 + 0.525 \approx 0.817 \\
F(2) &= 0.817 + 0.175 \approx 0.992 \\
F(3) &= 0.992 + 0.008 = 1
\end{aligned}
$$

- The expected value and variance of X.
  * Expected value

$$
\begin{aligned}
E[X] &= \sum_x xP(X=x) \\
&= 0\times P(X=0) + 1\times P(X=1) + 2\times P(X=2) + 3\times P(X=3) \\
&\approx 0.899
\end{aligned}
$$

  * Variance

$$
\begin{aligned}
\mathrm{Var}(X) &= \sum_x (x-E[X])^2 P(X=x) \\
&= (0-E[X])^2P(X=0) + (1-E[X])^2P(X=1) + \\
&\quad +(2-E[X])^2P(X=2) + (3-E[X])^2P(X=3) \\
&\approx 0.489
\end{aligned}
$$

- Answer first three points again, assuming that the three inspections were conducted with replacement.

To be solved

3 Engine Testing

At an airline maintenance facility, there are three identical aircraft engines of the same model that operate independently. The probability of each engine failing during a given time period is 0.1. Let X be the random variable representing the number of engines that are still operational at the end of this time period. Determine:

- The probability mass function of X.

$$P(X=0) = 1 \times 0.1 \times 0.1 \times 0.1 = 0.001$$
$$P(X=1) = 3 \times 0.9 \times 0.1 \times 0.1 = 0.027$$
$$P(X=2) = 3 \times 0.9 \times 0.9 \times 0.1 = 0.243$$
$$P(X=3) = 1 \times 0.1 \times 0.9 \times 0.9 = 0.729$$

- The cumulative distribution function of X.

$$F(0) = 0.001$$
$$F(1) = 0.028$$
$$F(2) = 0.271$$
$$F(3) = 1.000$$

- The expected value, mode, median, and variance of X.

$$\text{Mode} = 3$$

$$\text{Median} = 2 + \frac{3-2}{1-0.271}(0.5-0.271) \approx 2.31$$

$$E[X] = \sum_i xP(X=x)$$

$$\text{Var}(X) = \sum_i (x - E[X])^2 P(X=x)$$

4 Airline Booking Service

At an airline ticket counter, the time interval Δt (in minutes) between any two consecutive flight reservation calls is modeled by the following probability density function:

$$f(\Delta t) = \begin{cases} e^{-\Delta t}, & \text{if } \Delta t \geq 0, \\ 0, & \text{if } \Delta t < 0. \end{cases}$$

- Calculate $P(\Delta t > 2)$.

$$\begin{aligned} P(\Delta t > 2) &= \int_{2}^{+\infty} e^{-\Delta t} \, d\Delta t \\ &= \lim_{k \to +\infty} \int_{2}^{k} e^{-\Delta t} \, d\Delta t \\ &= \lim_{k \to +\infty} \left[-e^{-\Delta t} \right]_{2}^{k} \\ &= \lim_{k \to +\infty} \left(-e^{-k} + e^{-2} \right) \\ &= e^{-2} \end{aligned}$$

- Calculate $P(\Delta t > 3)$.

$$\begin{aligned} P(\Delta t > 2) &= \int_{3}^{+\infty} e^{-\Delta t} \, d\Delta t \\ &= e^{-3} \end{aligned}$$

- Calculate the conditional probability $P(\Delta t > 3 \mid \Delta t > 1)$.

$$\begin{aligned} P(\Delta t > 3 \mid \Delta t > 1) &= \frac{P(\Delta t > 3 \wedge \Delta t > 1)}{P(\Delta t > 1)} \\ &= \frac{\displaystyle\int_{3}^{+\infty} e^{-\Delta t} \, d\Delta t}{\displaystyle\int_{1}^{+\infty} e^{-\Delta t} \, d\Delta t} \\ &= \frac{e^{-3}}{e^{-1}} = e^{-2} \end{aligned}$$

- ***Extra***: Calculate the expected value

$$
\begin{aligned}
E[\Delta t] &= \int_{-\infty}^{+\infty} \Delta t e^{-\Delta t} \, d\Delta t \\
&= \int_{0}^{+\infty} \Delta t e^{-\Delta t} \, d\Delta t \\
&= \lim_{k \to +\infty} \int_{0}^{k} \Delta t e^{-\Delta t} \, d\Delta t
\end{aligned}
$$

$$\textbf{Using:} \int u dv = uv - \int v du$$

$$
\begin{aligned}
&= \lim_{k \to +\infty} \left(\left[-\Delta t e^{-\Delta t} \right]_0^k - \int_0^k -e^{-\Delta t} \, d\Delta t \right) \\
&= \lim_{k \to +\infty} \left(\left[-\Delta t e^{-\Delta t} \right]_0^k - \left[e^{-\Delta t} \right]_0^k \right) \\
&= \lim_{k \to +\infty} \left(-k e^{-k} - (e^{-k} - 1) \right) \\
&= 1
\end{aligned}
$$

5 Flight Altitude

Let X represent the altitude (in kilometers) of an aircraft during a specific flight phase. The probability density function of X is given by:

$$
f(x) = \begin{cases} k(9x - 6x^2 + x^3), & \text{if } 0 \leq x \leq 3, \\ 0, & \text{otherwise.} \end{cases}
$$

- Determine the value of k to ensure $f(x)$ is a valid probability density function.

Plot the function $f(x)$.

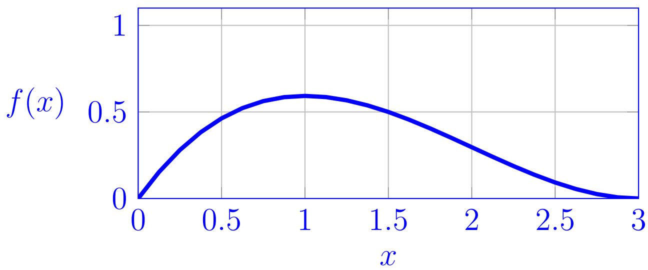
$$\int_{-\infty}^{+\infty} f(x)\,dx = 1$$

$$\int_0^3 k(9x - 6x^2 + x^3)\,dx = 1$$

$$\left[k\left(\frac{9x^2}{2} - \frac{6x^3}{3} + \frac{x^4}{4}\right)\right]_0^3 = 1$$

$$k = 1\bigg/\left[\frac{9x^2}{2} - \frac{6x^3}{3} + \frac{x^4}{4}\right]_0^3$$

$$k = 4/27$$



- Calculate the probabilities $P(X \leq 1.5)$, $P(X \geq 2.0)$ and $P(1.0 \leq X \leq 2.5)$.

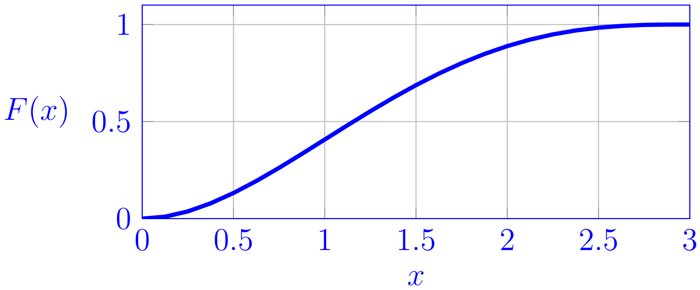
$$\begin{aligned} P(X \leq 1.5) &= \int_{-\infty}^{1.5} \frac{4}{27}(9x - 6x^2 + x^3)\,dx \\ &= \int_0^{1.5} \frac{4}{27}(9x - 6x^2 + x^3)\,dx \\ &\approx 0.688 \end{aligned}$$

$$\begin{aligned} P(X \geq 2.0) &= \int_2^{+\infty} \frac{4}{27}(9x - 6x^2 + x^3)\,dx \\ &= \int_2^3 \frac{4}{27}(9x - 6x^2 + x^3)\,dx \\ &\approx 0.111 \end{aligned}$$

$$
\begin{aligned}
P(1.0 \leq X \leq 2.5) &= \int_{1.0}^{2.5} \frac{4}{27}(9x - 6x^2 + x^3)\,dx \\
&\approx 0.576
\end{aligned}
$$

- Compute the cumulative distribution function $F(x)$, and plot it graphically.

$$
\begin{aligned}
F(x) = P(X \leq x) &= \int_{0}^{x} \frac{4}{27}(9x - 6x^2 + x^3)\,dx \\
&= \frac{4}{27}\left(\frac{9x^2}{2} - \frac{6x^3}{3} + \frac{x^4}{4}\right)
\end{aligned}
$$



6 Training Sessions

Consider a discrete random variable X, representing the number of successful landings by a pilot during a training session, with the following probability function:

$$
P(X = x) = \begin{cases} ax & \text{if } x \in \{1, 2, 3\}, \\ 0 & \text{otherwise,} \end{cases}
$$

where a is a real constant.

- Determine the value of a.

$$
\begin{aligned}
\sum_x P(X = x) &= 1 \\
P(X = 1) + P(X = 2) + P(X = 3) &= 1 \\
a + 2a + 3a &= 1 \\
a &= 1/6
\end{aligned}
$$

- Determine the cumulative distribution function (CDF) of X.

$$
\begin{aligned}
F(x) &= \sum_{i=1}^{x} P(X = x) \\
F(1) &= P(X = 1) = 1/6 \\
F(2) &= P(X = 1) + P(X = 2) = 1/2 \\
F(3) &= P(X = 1) + P(X = 2) + P(X = 3) = 1
\end{aligned}
$$

- Calculate the mode, median, and expected value of X.
  - Mode
  $$X = 3$$
  - Median
  $$X = 2$$
  - Expected Value

$$
\begin{aligned}
E[X] &= \sum_{i} xP(X = x) \\
&= 1 \times P(X = 1) + 2 \times P(X = 2) + 3 \times P(X = 3) \\
&= 1 \times 1/6 + 2 \times 1/3 + 3 \times 1/2 \\
&\approx 2.333
\end{aligned}
$$

7 Fuel Efficiency

An aircraft's fuel efficiency (in miles per gallon) can be considered a random variable X with the probability density function given by:

$$
f(x) = \begin{cases} \frac{3}{5} 10^{-5} x(100 - x), & \text{if } 0 \leq x \leq 100, \\ 0, & \text{otherwise.} \end{cases}
$$

Suppose the profit L obtained from operating the aircraft (per flight hour) depends on the fuel efficiency according to the relationship:

$$L = C_1 + C_2 X,$$

where C_1 and C_2 are constants.

- Calculate the expected value and variance of the profit per flight hour.
  - Expected Value

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{+\infty} x f(x)\, dx \\
&= \int_0^{100} x \left(\frac{3}{5} 10^{-5} x(100-x) \right)\, dx \\
&= \frac{3}{5} 10^{-5} \int_0^{100} 100x^2 - x^3\, dx \\
&= 50
\end{aligned}
$$

$$
\begin{aligned}
E[L] &= E[C_1 + C_2 X] \\
&= E[C_1] + E[C_2 X] \\
&= C_1 + C_2 E[X] \\
&= C_1 + 50C_2
\end{aligned}
$$

  - Variance

$$
\begin{aligned}
\mathrm{Var}(X) &= \int_{-\infty}^{+\infty} (x - E[X])^2 f(x)\, dx \\
&= \int_0^{100} (x-50)^2 \left(\frac{3}{5} 10^{-5} x(100-x) \right)\, dx \\
&= 500
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}(L) &= \mathrm{Var}(C_1 + C_2 X) \\
&= \mathrm{Var}(C_2 X) \\
&= C_2^2 \mathrm{Var}(X) \\
&= 500C_2^2
\end{aligned}
$$

8 Maintenance Process

Assume that, in a certain aviation maintenance process, the temperature of the engine oil recorded at the start of each shift in a specific aircraft follows a distribution with an expected value of 153°F and a standard deviation of 7°F.

- Calculate these two parameters of the temperature distribution when expressed in Celsius.

$$C = \frac{5}{9}(F - 32)$$

  - Expected Value

$$\begin{aligned} E[C] &= E\left[\frac{5}{9}(F - 32)\right] \\ &= E\left[\frac{5}{9}F\right] + E\left[-32 \times \frac{5}{9}\right] \\ &= \frac{5}{9}E\left[F\right] - \frac{160}{9} \\ &\approx 67.2\ ^{\circ}\mathrm{C} \end{aligned}$$

  - Variance

$$\begin{aligned} \mathrm{Var}(C) &= \mathrm{Var}\left(\frac{5}{9}(F - 32)\right) \\ &= \mathrm{Var}\left(\frac{5}{9}F\right) \\ &= \frac{25}{81}\mathrm{Var}(F) \end{aligned}$$

  - Standard Deviation

$$\begin{aligned} \sqrt{\mathrm{Var}(C)} &= \sqrt{\frac{25}{81}\mathrm{Var}(F)} \\ &\approx 3.89\ ^{\circ}\mathrm{C} \end{aligned}$$