Data Science in Aerospace

Jointly Distributed Random Variables

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ISEC Lisboa
ESCOLA DE GESTÃO, ENGENHARIA E AERONÁUTICA

1 Landing Performance

An aircraft performs three consecutive flights. Let X be the number of successful landings in the first two flights, and let Y be the number of unsuccessful landings in the last two flights. Assume that both events have equal probability.

- Define the joint probability function of the variables X and Y.

$$X: \text{ number of successful landings in the first two flights}$$

$$Y: \text{ number of unsuccessful landings in the last two flights}$$

$$P(X, Y)$$

$$\begin{array}{lll} P(0,0) = 0 & P(1,0) = 1/8 & P(2,0) = 1/8 \\ P(0,1) = 1/8 & P(1,1) = 2/8 & P(2,1) = 1/8 \\ P(0,2) = 1/8 & P(1,2) = 1/8 & P(2,2) = 0 \end{array}$$

- Define the conditional probability function of Y, given that $X = 1$.

$$
\begin{aligned}
P_{Y|X=1}(Y=0) &= \frac{P(Y=0,X=1)}{P(X=1)} = \frac{1/8}{1/2} = 1/4 \\
P_{Y|X=1}(Y=1) &= \frac{P(Y=1,X=1)}{P(X=1)} = \frac{2/8}{1/2} = 1/2 \\
P_{Y|X=1}(Y=2) &= \frac{P(Y=2,X=1)}{P(X=1)} = \frac{1/8}{1/2} = 1/4
\end{aligned}
$$

- Calculate the correlation coefficient ρ_{XY}.

$$
\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}
$$

$$
\begin{aligned}
\mu_X = \sum_x x P_X(x) &= 0 \times (0 + 1/8 + 1/8) \\
&+ 1 \times (1/8 + 2/8 + 1/8) \\
&+ 2 \times (1/8 + 1/8 + 0) = 1 \\
\mu_Y = \sum_y y P_Y(y) &= 1
\end{aligned}
$$

$$
\begin{aligned}
\sigma_X^2 &= \sum_x (x-\mu_X)^2 P_X(x) = 1/\sqrt{2} \\
\sigma_Y^2 &= \sum_y (y-\mu_Y)^2 P_Y(y) = 1/\sqrt{2}
\end{aligned}
$$

Test if X and Y are independent

$$
P_{X,Y}(2,0) = P_X(2)P_Y(0) \Leftrightarrow 1/8 = 1/16 \Leftrightarrow \text{They are not independent}
$$

$$
\mathrm{Cov}(X,Y) = \sum_x \sum_y (x-\mu_X)(y-\mu_Y) P_{XY}(X,Y) = -1/4
$$

$$
\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} = \frac{-1/4}{1/\sqrt{2}} \approx -0.35
$$

2 Departures

An airport monitors the number of flights departing daily from two airlines, Airline X and Airline Y. The joint probability function of the number of flights departing daily is as follows:

X/Y	0	1	2
0	0.12	0.25	0.13
1	0.05	0.30	0.01
2	0.03	0.10	0.01

- Calculate the marginal probability functions of X and Y.
- Calculate the marginal distribution function of X.
- Calculate the probability that, on a given day, Airline Y has more departing flights than Airline X.
- Determine the expected value and variance of the total number of flights departing daily.

3 Flight Parameters

Let X and Y represent random variables related to aviation, where X denotes the wind speed (in knots) and Y denotes the altitude level (in thousands of feet). Their joint probability function is given by:

X/Y	-1	0	1
-1	0	$1/4$	0
0	$1/4$	0	$1/4$
1	0	$1/4$	0

- Show that $\mathrm{Cov}(X, Y) = 0$ but that X and Y are not independent.

4 Pilot Certification

To qualify for a pilot certification, a candidate must complete two independent tests, A (theoretical test) and B (practical flight test). The performance in each test is classified as insufficient (0), sufficient (1), or excellent (2). The probability of the candidate achieving scores of 0, 1, or 2 in tests A and B is provided in the following table:

Score	Test A	Test B
0	0.2	0.2
1	0.5	0.6
2	0.3	0.2

Consider the random pair (X, Y), where X is the absolute difference between the scores obtained in tests A and B and Y is the sum of the scores obtained in tests A and B.

- Determine the joint probability function of the random pair (X, Y).

$$X: \text{ is the absolute difference between the scores obtained in tests } A \text{ and } B$$

$$Y: \text{ sum of the scores obtained in tests } A \text{ and } B$$

$$P(X, Y)$$

$$\begin{aligned}
P(0,0) &= 0.2 \times 0.2 = 0.04 \\
P(0,1) &= 0 \\
P(0,2) &= 0.5 \times 0.6 = 0.3 \\
P(0,3) &= 0 \\
P(0,4) &= 0.3 \times 0.2 = 0.06
\end{aligned}$$

$$\begin{aligned}
P(1,0) &= 0 \\
P(1,1) &= 0.2 \times 0.6 + 0.5 \times 0.2 = 0.22 \\
P(1,2) &= 0 \\
P(1,3) &= 0.5 \times 0.2 + 0.3 \times 0.6 = 0.28 \\
P(1,4) &= 0
\end{aligned}$$

$$P(2,0) = 0$$
$$P(2,1) = 0$$
$$P(2,2) = 0.2 \times 0.2 + 0.3 \times 0.2 = 0.1$$
$$P(2,3) = 0$$
$$P(2,4) = 0$$

X/Y	0	1	2	3	4	P_X
0	0.04	0	0.30	0	0.06	0.4
1	0	0.22	0	0.28	0	0.5
2	0	0	0.1	0	0	0.1
P_Y	0.04	0.22	0.4	0.28	0.06	

- Determine the marginal probability functions of X and Y.

$$P_X(X=0) = 0.4$$
$$P_X(X=1) = 0.5$$
$$P_X(X=3) = 0.1$$

$$P_Y(Y=0) = 0.04$$
$$P_Y(Y=1) = 0.22$$
$$P_Y(Y=2) = 0.4$$
$$P_Y(Y=3) = 0.28$$
$$P_Y(Y=4) = 0.06$$

- Determine the cumulative distribution function of X.

$$F_X(0) = 0.4$$
$$F_X(1) = 0.9$$
$$F_X(2) = 1.0$$

- Determine the conditional probability function of X given that $Y = 2$.

$$P_{X|Y=2}(X=0) = \frac{P(X=0, Y=2)}{P(Y=2)} = \frac{0.3}{0.4} = 0.75$$
$$P_{X|Y=2}(X=1) = \frac{P(X=1, Y=2)}{P(Y=2)} = \frac{0}{0.4} = 0$$
$$P_{X|Y=2}(X=2) = \frac{P(X=2, Y=2)}{P(Y=2)} = \frac{0.1}{0.4} = 0.25$$

- State, with justification, whether X and Y are independent.
  - Test if X and Y are independent

$$P_{XY}(0,0) = P_X(0)P_Y(0) \Leftrightarrow 0.04 = 0.016 \Leftrightarrow \text{They are not independent}$$

- Calculate all conditional probability functions of Y given X.

$$\begin{aligned}
P_{Y|X}(Y=1) &= \frac{P(Y=1, X=0)}{P_X(X=0)} \\
P_{Y|X}(Y=1) &= \frac{P(Y=1, X=1)}{P_X(X=1)} \\
P_{Y|X}(Y=1) &= \frac{P(Y=1, X=2)}{P_X(X=2)} \\
&\vdots \\
P_{Y|X}(Y=4) &= \frac{P(Y=4, X=2)}{P_X(X=2)}
\end{aligned}$$

- Calculate the expected value $E[Y|X=2]$ and variance $\mathrm{Var}[Y|X=2]$.

$$E[Y|X=2] = \sum_y y P_Y(Y=y \mid X=2)$$

$$\mathrm{Var}(Y|X=2) = \sum_y (y - \mu_{Y|X=2})^2 P_Y(Y=y \mid X=2)$$

- Calculate $F_{Y|X=0}(y)$.

$$\begin{aligned}
F_{Y|X=0}(0) &= 0.04 \\
F_{Y|X=0}(1) &= 0.04 \\
F_{Y|X=0}(2) &= 0.34 \\
F_{Y|X=0}(3) &= 0.34 \\
F_{Y|X=0}(4) &= 0.40
\end{aligned}$$

- Calculate the probability that $Y = 2$ given that $X \cdot Y = 0$.

$$P = \frac{P(0,2)}{P(X = 0 \vee Y = 0)} = 0.75$$

- Calculate the probability that $X + Y$ is odd.

$$P = 0$$

5 Cruise Flight

Consider two random variables X and Y that represent the altitude (in kilometers) and airspeed (in Mach) of an aircraft, respectively. The joint probability density function is given by:

$$f(x,y) = \begin{cases} 2, & \text{if } (x,y) \in D, \\ 0, & \text{if } (x,y) \notin D, \end{cases}$$

where $D = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq x\}$.

- Calculate:
  - The marginal distributions of the altitude (X) and airspeed (Y).

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f(x,y)\, dy \\ &= \int_0^x 2\, dy \\ &= [2y]_0^x \\ &= 2x \end{aligned} \qquad \begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)\, dx \\ &= \int_y^1 2\, dx \\ &= [2x]_y^1 \\ &= 2 - 2y \end{aligned}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx \qquad\qquad F_Y(y) = \int_{-\infty}^{y} f_Y(y)\, dy$$
$$= \int_0^x 2x\, dx \qquad\qquad = \int_0^y 2 - 2y\, dy$$
$$= \left[x^2\right]_0^x \qquad\qquad = \left[2y - y^2\right]_0^y$$
$$= x^2 \qquad\qquad = 2y - y^2$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \qquad F_Y(y) = \begin{cases} 0 & y < 0 \\ 2y - y^2 & 0 \le y \le 1 \\ 1 & y > 1 \end{cases}$$

– The expected values of the altitude (X) and airspeed (Y).

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx \qquad\qquad E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy$$
$$= \int_0^1 2x^2\, dx \qquad\qquad = \int_0^1 2y - 2y^2\, dy$$
$$= \left[\frac{2x^3}{3}\right]_0^1 \qquad\qquad = \left[\frac{2y^2}{2} - \frac{2y^3}{3}\right]_0^1$$
$$= 2/3 \qquad\qquad = 1/3$$

– The conditional distributions of the altitude (X), given airspeed $Y = y$, and of the airspeed (Y), given altitude $X = x$.

$$f_{X|Y=y}(x \mid y) = \frac{f(x \mid y)}{f_Y(y)} \qquad\qquad f_{Y|X=x}(y \mid x) = \frac{f(y \mid x)}{f_X(x)}$$
$$= \frac{1}{1-y} \qquad\qquad = \frac{1}{x}$$
$$(y \le x \le 1) \qquad\qquad (0 \le y \le x)$$

$$F_{X|Y=y}(x \mid y) = \int_y^x \frac{1}{1-y} dx \qquad\qquad F_{Y|X=x}(y \mid x) = \int_0^y \frac{1}{x} dy$$
$$= \frac{x-y}{1-y} \qquad\qquad = \frac{y}{x}$$
$$(y \le x \le 1) \qquad\qquad (0 \le y \le x)$$

– The covariance between the altitude (X) and airspeed (Y).

$$
\begin{aligned}
\text{Cov}(X, Y) &= \iint_{\mathbb{R}^2} (x - \mu_X)(y - \mu_Y) f(x, y)\, dxdy \\
&= 2 \int_0^1 \int_0^x \left(x - \frac{2}{3}\right)\left(y - \frac{1}{3}\right)\, dydx \\
&\approx 0.0278
\end{aligned}
$$

6 Drone Operations

Consider the joint probability density function of two random variables X and Y, where X represents the time (in hours) a drone spends flying, and Y represents the fuel consumption (in liters) during the flight:

$$
f(x, y) = \begin{cases} 6(1 - x - y), & \text{if } 0 \leq y \leq 1 - x \text{ and } x > 0, \\ 0, & \text{otherwise.} \end{cases}
$$

- Are the variables X (flight time) and Y (fuel consumption) independent? Justify your answer.
- Compute the cumulative distribution function (CDF) of the random variable X.
- Determine the conditional density function $f_{X|Y=y}(x)$.
- Calculate $P(X < 1/4 \mid Y = 1/2)$, i.e., the probability that the drone's flight time is less than 1/4 hours given that its fuel consumption is 1/2 liters.
- Calculate $P(X < 3/4 \mid Y > 1/2)$, i.e., the probability that the drone's flight time is less than 3/4 hours given that its fuel consumption exceeds 1/2 liters.

7 Flight Operations

Two airplanes are scheduled to arrive at an airport between 14:00 and 15:00. It is understood that no airplane will wait more than 15 minutes for the other to arrive. Assume that all arrival times within this interval are equally likely.

- What is the probability that the two airplanes will meet at the airport?

8 Wing Design

The wingspan W and the chord length c of an aircraft are random variables with the following parameters:

$$W : \mu_W = 36\,\mathrm{m},\ \sigma_W = 0.5\,\mathrm{m}$$
$$L : \mu_c = 4\,\mathrm{m},\ \sigma_c = 0.1\,\mathrm{m}$$

- Assuming W and c are independent and that the wing is rectangular, estimate the expected value and the standard deviation for the aircraft's wing area.

9 Manufacturing Process

In an aircraft manufacturing process, oscillation systems are produced for a specific type of aircraft. These systems include a damping mechanism with a resistance R and a capacitor with capacitance C. The oscillation period T of the system depends on R and C according to the following relationship:

$$T = \frac{R^3 C^3}{100}.$$

Assume that R and C are random variables with the following parameters:

$$R: \quad \mu_R = 10^6, \quad \sigma_R = 0.03\mu_R,$$

$$C: \quad \mu_C = 10^{-6}, \quad \sigma_C = 0.05\mu_C.$$

Suppose that the resistance and the capacitor incorporated into each oscillation system are selected independently of one another.

- Estimate the expected value and the standard deviation of the oscillation period for these systems.

$$\begin{aligned}
T(R,C) \approx\ & T(\mu_R,\mu_C) + \frac{\partial T}{\partial R}(\mu_R,\mu_C)(R-\mu_R) + \frac{\partial T}{\partial C}(\mu_R,\mu_C)(C-\mu_C) \\
&= \frac{(10^6 10^{-6})^3}{100} + \frac{3\mu_R^3\mu_C^2}{100}(R-10^6) + \frac{3\mu_C^2\mu_R^3}{100}(C-10^{-6}) \\
&= -\frac{1}{20} + 3\times 10^{-8}R + 3\times 10^4 C
\end{aligned}$$

$$E[T] \approx E\left[-\frac{1}{20} + 3 \times 10^{-8} R + 3 \times 10^{4} C\right] = \frac{1}{100}$$

$$\mathrm{Var}(Z) = \mathrm{Var}(X) + \mathrm{Var}(Y) + \mathrm{Cov}(X, Y)$$

$$\begin{aligned}
\mathrm{Var}(T) &\approx \mathrm{Var}\left(-\frac{1}{20} + 3 \times 10^{-8} R + 3 \times 10^{4} C\right) \\
&= \left(3 \times 10^{-8}\right)^2 \mathrm{Var}(R) + \left(3 \times 10^{4}\right)^2 \mathrm{Var}(C) \\
&\approx 3.06 \times 10^{-6}\ \mathrm{s}^2
\end{aligned}$$