Computational Applied Mathematics

Exercices

ESCOLA DE GESTÃO, ENGENHARIA E AERONÁUTICA

Taylor Series

1. Determine the fourth-order Taylor polynomial of $f(x) = \cos(x)$ centered at 0.
   SOLUTION HERE
2. Consider the function $f(x) = \sin x$. Approximate $f(\pi/8)$ and $f(10\pi/9)$ using the fifth-order Taylor polynomial centered at 0.
   SOLUTION HERE
3. What is the second term in the Taylor series of $\sqrt[4]{4x-1}$ about 4.25?
   SOLUTION HERE
4. In the Taylor series $\left(\text{about } \pi/4\right)$ for the function $\sin x + \cos x$, find the third nonzero term.
   SOLUTION HERE
5. In the Taylor series for the function $3x^2 - 7 + \cos x$ $\left(\text{expanded in power of } x\right)$, what is the coefficient of x^2?
   SOLUTION HERE

Numerical Differentiation

1. Calculate the first-order derivative of the function $g(x) = e^{\sin(x)}$ at the point $x = 0.5$ using first-order finite differences (forward, backward, and central) with $h = 0.01$.

   SOLUTION HERE

2. Calculate the second-order derivative of the function $g(x) = e^{\cos(x)}$ at the point $x = 0.5$ using second-order finite differences (forward, backward, and central) with $h = 0.01$.

   SOLUTION HERE

3. The distance traveled in meters by a rocket at each second is given by the following values:

t[s]	0	1	2	3	4	5
y[m]	0.0	2.5	7.8	18.2	51.9	80.3

   Use numerical differentiation to approximate the velocity and acceleration at each point.

   SOLUTION HERE

4. Determine the second-order Taylor polynomial of $f(x) = x^x \cos(e^{x^2}) + \ln(\sin(x^3))$ centered at 1. Use numerical differentiation instead of analytical differentiation.

   SOLUTION HERE

5. Consider the function $f(x) = \sin(x)$ at $x_0 = 1.0$, $x_1 = 1.4$, and $x_2 = 2.0$. Approximate the derivative $f'(x_1)$ at x_1 using the divided difference formula. Compare this approximation with the exact derivative.

$$f(x_0) \approx f(x_1) + f'(x_1)(x_0 - x_1)$$
$$f(x_2) \approx f(x_1) + f'(x_1)(x_2 - x_1)$$
$$f(x_0) + f(x_2) = 2f(x_1) + 2f'(x_1)(x_0 - x_1)(x_2 - x_1)$$
$$f'(x_1) = \frac{f(x_0) - 2f(x_1) + f(x_2)}{2(x_0 - x_1)(x_2 - x_1)}$$

$$f'(x_1) \approx \frac{(x_1 - x_2)^2\big(f(x_0) - f(x_1)\big) + (x_1 - x_0)^2\big(f(x_2) - f(x_1)\big)}{(x_1 - x_0)(x_1 - x_2)(x_2 - x_0)}$$

Zeros

1. Using the bisection method, determine the point of intersection of the curves given by

$$y = x^3 - 2x + 1 \quad \text{and} \quad y = x^2.$$

   SOLUTION HERE

2. Find a root of the equation $6(e^x - x) = 6 + 3x^2 + 2x^3$ between -1 and $+1$ using the bisection method.
   SOLUTION HERE

3. Find a root of the equation $6(e^x - x) = 6 + 3x^2 + 2x^3$ between -1 and $+1$ using the false position method.
   SOLUTION HERE

4. Use the false position method to find a zero of the equation

$$\lambda \cosh\left(\frac{50}{\lambda}\right) = \lambda + 10$$

   SOLUTION HERE

5. If Newton's method is used on $f(x) = x^3 - x + 1$ starting with $x_0 = 1$, what will x_1 be?
   SOLUTION HERE

6. Using the Newton's method, locate the root of $f(x) = e^{-x} - \cos x$ that is nearest $\pi/2$.
   SOLUTION HERE

7. Carry out three iterations of Newton's method using $x_0 = 1$ and $f(x) = 3x^3 + x^2 - 15x + 3$.
   SOLUTION HERE

8. If the secant method is used on $f(x) = x^5 + x^3 + 3$ and if $x_{n-2} = 0$ and $x_{n-1} = 1$, what is x_n?
   SOLUTION HERE

9. Use the secant method to find the zero near -0.5 of $f(x) = e^x - 3x^2$.
   SOLUTION HERE

10. Knowing that $f(x) = \dfrac{\cos(\ln x)}{e^x}$, use the secant method to find the zero near 0.25.
    SOLUTION HERE

Extrema of Functions

1. Use the golden section algorithm to determine the minimum of $f(x) = 2x^3 - 9x^2 + 12x + 2$ on $[0, 3]$.
   SOLUTION HERE
2. Consider the function $f(x) = -1.5x^6 - 2.4x^4 + 12x$. Estimate the value of $x \in [0, 2]$ that maximizes $f(x)$ by applying the quadratic interpolation method three times with $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$.
   SOLUTION HERE
3. Find an estimate for the minimum of $f(x) = x + \frac{1}{x}$ using the quadratic interpolation method with $x_0 = 0.1$, $x_1 = 0.5$, $x_2 = 1.0$.
   SOLUTION HERE
4. Find an estimate of the maximizer of $f(x) = 2x^3 - 9x^2 + 12x$ in $[0, 2]$ using Newton's method with $x_0 = 2$ and iterating 3 times.
   SOLUTION HERE
5. Using the Newton's method, locate the minium of $f(x) = 0.01x^2 + \frac{500000}{0.25x^2}$ starting with $x_0 = 10$.
   SOLUTION HERE

Systems of Equations

1. Calculate the ℓ_1, ℓ_2 and ℓ_∞ norms of

$$\begin{bmatrix}10 & 7 & 15\end{bmatrix}^T$$

   SOLUTION HERE

2. Calculate the ℓ_1, ℓ_F and ℓ_∞ norms of

$$\begin{bmatrix}3 & 4 & 3\\ 1 & 5 & -1\\ 6 & 3 & 7\end{bmatrix}$$

   SOLUTION HERE

3. Solve the system of equations

$$\begin{cases}3x_1 + 4x_2 + 3x_3 = 10\\ x_1 + 5x_2 - x_3 = 7\\ 6x_1 + 3x_2 + 7x_3 = 15\end{cases},$$

   using naive Gaussian elimination, Gaussian elimination with partial and complete pivoting.

   SOLUTION HERE

4. Solve the system of equations

$$\begin{bmatrix}1 & -1 & 2 & 1\\ 3 & 2 & 1 & 4\\ 5 & 8 & 6 & 3\\ 4 & 2 & 5 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix}1\\ 1\\ 1\\ -1\end{bmatrix},$$

   using the Gauss-Jordan elimination.

   SOLUTION HERE

5. Let

$$\begin{bmatrix}4 & 1 & -1\\ 1 & 2 & 4\\ 1 & 5 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}10\\ 9\\ 11\end{bmatrix}.$$

   Carry out a number of iterations of the Jacobi iteration, starting with the $[1, 1, 1]^T$ initial vector. Start by verifying diagonal dominance.

   SOLUTION HERE

6. Let

$$\begin{bmatrix} -2 & 1 & 0 \\ -1 & 3 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ -5 \end{bmatrix}.$$

Carry out a number of iterations of the Gauss-Seidel iteration, starting with the zero initial vector. Start by verifying diagonal dominance.

SOLUTION HERE

7. Use the Newton's method to solve the system of equations

$$\begin{cases} x + y + z = 3 \\ x^2 + y^2 + z^2 = 5 \\ e^x + xy - xz = 1 \end{cases},$$

starting with $[0.1, 1.2, 2.5]^T$.

SOLUTION HERE

8. Solve this pair of simultaneous nonlinear equations by first eliminating y and then solving the resulting equation in x by Newton's method. Start with the initial value $x_0 = 1.0$.

$$\begin{cases} x^3 - 2xy + y^7 - 4x^3y = 5 \\ y \sin x + 3x^2y + \tan x = 4 \end{cases}$$

$$y \sin x + 3x^2y + \tan x = 4 \Rightarrow y = \frac{4 - \tan x}{\sin x + 3x^2}$$

$$x^3 - 2xy + y^7 - 4x^3y = 5 \Rightarrow$$

$$\Rightarrow x^3 - \frac{2x(4 - \tan x)}{\sin x + 3x^2} + \left(\frac{4 - \tan x}{\sin x + 3x^2}\right)^7 - \frac{4x^3(4 - \tan x)}{\sin x + 3x^2} - 5 = 0$$

Interpolation & Approximation

1. Use the Lagrange interpolation process to obtain a polynomial of least degree that assumes these values:

x	0	2	3	4
y	7	11	28	63

SOLUTION HERE

2. Use the Newton interpolation process to obtain a polynomial of least degree that assumes these values:

x	1	3	-2	4	5
y	2	6	-1	-4	2

SOLUTION HERE

3. Determine the natural cubic spline that interpolates the function at the given points.

x	0	0.5	0.7	1.0
y	0	0.6	1.2	2.2

SOLUTION HERE

4. Using the least-square method, fit the data in the table

x	0.50	0.55	0.60	0.65	0.70	0.75	0.80
y	1.2	1.0	0.7	0.4	0.1	-0.2	-0.6

by a parabola.
SOLUTION HERE

5. Fit a function of the form $y = a \ln x + b \cos x + ce^x$ to the following table values

x	0.24	0.65	0.95	1.24	1.73	2.01	2.23	2.52	2.77	2.99
y	0.23	-0.26	-1.10	-0.45	0.27	0.10	-0.29	0.24	0.56	1.00

SOLUTION HERE

6. Using the least-square method, fit the data in the table

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
y	0.6	1.1	1.6	1.8	2.0	1.9	1.7	1.3

by a function $y = a \sin(bx)$. Do it with and without linearization.
SOLUTION HERE

Numerical Integration

1. What is the numerical value of the composite trapezoid rule applied to the reciprocal function $f(x) = x^{-1}$ using the points 1, 4/3, 2?
   SOLUTION HERE
2. Approximate $\int_0^2 2^x\ dx$ using the composite trapezoid rule with $h = 1/2$.
   SOLUTION HERE
3. Approximate $\int_1^2 f(x)\ dx$ given the table of values

x	1	5/4	3/2	7/4	2
$f(x)$	10	8	7	6	5

   Compute an estimate by the composite trapezoid rule.
   SOLUTION HERE
4. Use the composite trapezoidal and Simpson's rules with 4 sub-intervals to approximate the following integral and determine an upper bound for the error.

$$\int_1^2 x \ln(x)\, dx$$

   SOLUTION HERE
5. Consider the integral

$$\int_1^{1.6} \frac{2x}{x^2 - 4}\, dx$$

   Apply all of the Newton-Cotes closed rules.
   SOLUTION HERE
6. Consider the integral

$$\int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}}\, dx$$

   Because it has singularities at the endpoints of the interval $[-1, 1]$, closed rules cannot be used. Apply all of the Newton-Cotes open rules. Compare and explain these numerical results to the true solution, which is

$$\int_{-1}^{1} (1 - x^2)^{-1/2}\, dx = \arcsin x \Big|_{-1}^{1} = \pi.$$

   SOLUTION HERE

Ordinary Differential Equations

1. Consider the initial value problem

$$\begin{cases} y' = -ty \\ y(0) = 1 \end{cases}, \quad t \in [0, +\infty[. \quad (\text{solution} : y(t) = e^{-t^2/2}).$$

   Calculate an approximation for the solution at $t = 0.5$, using the Euler method with step $h = 0.1$.
   SOLUTION HERE

2. Given the initial value problem

$$\begin{cases} y' = \dfrac{y}{1+t^2} \\ y(0) = 1 \end{cases}, \quad 0 \le t \le 1 \quad \left(\text{solution} : y(t) = e^{\arctan(t)}\right).$$

   Use the second-order Taylor method to compute an approximate value y_5 for $y(1)$.
   SOLUTION HERE

3. Using Euler's method, compute an approximate value for $x(2)$ for the differential equation

$$x' = 1 + x^2 + t^3,$$

   with the initial value $x(1) = -4$ using 100 steps.
   SOLUTION HERE

4. Consider the ordinary differential equation

$$\begin{cases} x' = t^2 + tx' - 2xx' \\ x(0) = 1 \end{cases}.$$

   Take three steps using the Runge-Kutta method of order 2 with step $h = 0.1$.
   SOLUTION HERE

5. Given the initial value problem

$$\begin{cases} y' = e^t y^2 + e^3 \\ y(2) = 4 \end{cases}.$$

   Use the fourth-order Runge-Kutta method to compute an approximate value y_5 for $y(5)$.
   SOLUTION HERE

6. Find approximations for the solution of the initial value problem

$$\begin{cases} y' = -ty^2 \\ y(1) = 2 \end{cases},$$

   for $t \in [1, 2]$, with $h = 0.2$. Use the 2^{th} and 4^{th} order Adams-Bashforth-Moulton predictor-corrector method.
   SOLUTION HERE

Systems of Ordinary Differential Equations

1. Given the system

$$\begin{cases} x' = 2x - 3y \\ y' = y - 2x \\ x(0) = 8, \quad y(0) = 3 \end{cases},$$

   apply Euler's method to estimate $x(0.1)$ and $y(0.1)$ with two integration steps.
   SOLUTION HERE

2. Use the 2^{th} order Runge-Kutta method, with $h = 0.05$, to estimate the values $y(0.1)$ and $x(0.1)$ of the particular solution of

$$\begin{cases} x' = 2t - 3y \\ y' = t - x^2 \end{cases},$$

   with $x(0) = 2$ and $y(0) = 1$.
   SOLUTION HERE

3. Convert the third-order differential equation

$$y''' + ty'' - yy' + y^2 = \cos(t)$$

   to a system of first-order differential equations with initial conditions

$$y(1.2) = -1, \quad y'(1.2) = 0, \quad y''(1.2) = 1.$$

   SOLUTION HERE

4. Determine an approximate value for $y(0.5)$ using the Euler and second-order Taylor methods for the initial value problem

$$y'' + 2y' + 5y = e^{-t}\sin(t), \quad y(0) = 0, \quad y'(0) = 1$$

   SOLUTION HERE

5. Determine an approximate value for $y(0.5)$ using the second-order Adams-Bashforth-Multon method for the initial value problem

$$y'' - 3y' + 2y = 4e^{2t}, \quad y(0) = -3, \quad y'(0) = 5$$

   SOLUTION HERE